CALCULUS I/MATH 150
SHANNON GRACEY

π 100 POINTS POSSIBLE

π YOUR WORK MUST SUPPORT YOUR ANSWER FOR FULL CREDIT TO BE AWARDED

π YOU MAY USE A TI-83/84/85/86 CALCULATOR

π PROVIDE EXACT ANSWERS UNLESS OTHERWISE INDICATED

ONCE YOU BEGIN THE EXAM, YOU MAY NOT LEAVE THE PROCTORING CENTER UNTIL YOU ARE FINISHED…THIS MEANS NO BATHROOM BREAKS!

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(64 POINTS) Problems 1-8. Evaluate the definite integrals and find the indefinite integrals: Each question is worth 8 points. EXACT ANSWERS ONLY!!!

1. $\int_2^6 |x-3|\,dx$

$$|x-3| = \begin{cases} -(x-3), & x<3 \\ x-3, & x\ge 3 \end{cases}$$

$$= \int_2^3 -(x-3)\,dx + \int_3^6 (x-3)\,dx$$

$$= -\left(\tfrac{1}{2}x^2 - 3x\right)\Big|_{x=2}^{x=3} + \left(\tfrac{1}{2}x^2 - 3x\right)\Big|_{x=3}^{x=6}$$

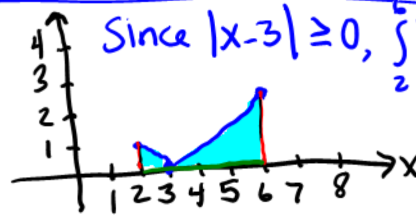
$$= -\left[\left(\tfrac{1}{2}(3)^2 - 3(3)\right) - \left(\tfrac{1}{2}(2)^2 - 3(2)\right)\right] + \left[\left(\tfrac{1}{2}(6)^2 - 3(6)\right) - \left(\tfrac{1}{2}(3)^2 - 3(3)\right)\right]$$

$$= -\left[\left(\tfrac{9}{2} - 9\right) - (2-6)\right] + \left[(18-18) - \left(\tfrac{9}{2} - 9\right)\right]$$

$$= -\left[-\tfrac{9}{2} - (-4)\right] + \left[0 - \left(-\tfrac{9}{2}\right)\right]$$

$$= -\left(-\tfrac{1}{2}\right) + \tfrac{9}{2} \rightarrow = \boxed{5}$$

Or... Since $|x-3| \ge 0$, $\int_2^6 |x-3|\,dx =$ area of the 2 triangles.



So...

$A = \tfrac{1}{2}(1)(1) + \tfrac{1}{2}(3)(3)$

$A = \tfrac{1}{2} + \tfrac{9}{2}$

$A = 5$

$$\boxed{\int_2^6 |x-3|\,dx = 5}$$

2. $\int \dfrac{2\theta^2}{\sin^2 \theta^3}\,d\theta$

$u = \theta^3$

$\dfrac{du}{d\theta} = 3\theta^2$

$d\theta = \dfrac{du}{3\theta^2}$

$$= \int 2\theta^2 \csc^2 \theta^3\, d\theta$$

$$= \int 2\cancel{\theta^2} \csc^2 u \,\frac{du}{3\cancel{\theta^2}}$$

$$= \frac{2}{3}\int \csc^2 u\,du$$

$$= \frac{2}{3}(-\cot u) + C$$

$$= \boxed{-\frac{2}{3}\cot \theta^3 + C}$$

3. $\int \frac{x}{\sqrt{1-x}}dx = \int x(1-x)^{-1/2}\,dx$

$u = 1-x \rightarrow x = 1-u$

$\frac{du}{dx} = -1$

$dx = -du$

$= \int x(u)^{-1/2}(-du)$

$= -\int (1-u)u^{-1/2}\,du$

$= -\int (u^{-1/2} - u^{1/2})\,du$

$= -\left(\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right) + C$

$= \boxed{-2(1-x)^{1/2} + \frac{2}{3}(1-x)^{3/2} + C}$

4. $\int (1+x^2)^3\,dx$

$(1+x^2)^3 = (1)^3(x^2)^0 + 3(1)^2(x^2)^1 + 3(1)^1(x^2)^2 + (1)^0(x^2)^3$

$(1+x^2)^3 = 1 + 3x^2 + 3x^4 + x^6$

$= \int (1 + 3x^2 + 3x^4 + x^6)\,dx$

$= x + \frac{3x^3}{3} + \frac{3x^5}{5} + \frac{x^7}{7} + C$

$= \boxed{x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7 + C}$

5. $\int \cos^2 5x\,dx = \int \frac{1+\cos[2(5x)]}{2}\,dx$

$\cos^2 A = \frac{1+\cos 2A}{2}$

$= \frac{1}{2}\int (1 + \cos 10x)\,dx$

$u = 10x$

$\frac{du}{dx} = 10$

$dx = \frac{du}{10}$

$= \frac{1}{2}\left[\int 1\,dx + \int \cos 10x\,dx\right]$

$= \frac{1}{2}\left[x + \int \cos u\left(\frac{du}{10}\right)\right]$

$= \frac{1}{2}\left[x + \frac{1}{10}\sin u\right] + C \rightarrow = \boxed{\frac{1}{2}x + \frac{1}{20}\sin 10x + C}$

6. $\int\left(\frac{4x+x^{3/4}}{x^{1/4}}\right)dx = \int\left(\frac{4x}{x^{1/4}}+\frac{x^{3/4}}{x^{1/4}}\right)dx$

$= \int\left(4x^{3/4}+x^{1/2}\right)dx$

$= \frac{4x^{7/4}}{7/4}+\frac{x^{3/2}}{3/2}+C$

$= \boxed{\frac{16}{7}x^{7/4}+\frac{2}{3}x^{3/2}+C}$

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7. $\int_3^5 \frac{x^3+1}{x+1}dx = \int_3^5 \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}}dx$

$= \int_3^5 (x^2-x+1)dx$

$= \left(\frac{x^3}{3}-\frac{x^2}{2}+x\right)\Big|_{x=3}^{x=5}$

$= \left[\left(\frac{125}{3}-\frac{25}{2}+5\right)-\left(\frac{27}{3}-\frac{9}{2}+3\right)\right]$

$= \frac{98}{3}-\frac{33}{2}+5-3$

$= \frac{196-99}{6}+2$

$= \frac{97}{6}+\frac{12}{6} \to = \boxed{\frac{109}{6}}$

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8. $\int_{\pi/4}^{\pi/3} \tan^3 x \sec^2 x dx = \int_1^{\sqrt{3}} u^3 \cancel{\sec^2 x}\left(\frac{du}{\cancel{\sec^2 x}}\right)$

$= \int_1^{\sqrt{3}} u^3 du$

$= \frac{u^4}{4}\Big|_{u=1}^{u=\sqrt{3}}$

$= \frac{(\sqrt{3})^4-(1)^4}{4}$

$= \boxed{2}$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$u(\pi/4) = \tan \pi/4 = 1$

$u(\pi/3) = \tan \pi/3 = \sqrt{3}$

9. (5 POINTS) Find the average value of the function $f(x)=\frac{4}{x^2}$ on the interval $[1,4]$.

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$a=1,\ b=4$

$f(x)=4x^{-2}$

$$= \frac{1}{4-1}\int_1^4 4x^{-2}\,dx$$

$$= \frac{4}{3}\int_1^4 x^{-2}\,dx$$

$$= \frac{4}{3}\left.\frac{x^{-1}}{-1}\right|_{x=1}^{x=4}$$

$$= -\frac{4}{3}\left(\frac{1}{4}-\frac{1}{1}\right)$$

$$= -\frac{4}{3}\left(-\frac{3}{4}\right)$$

$$= \boxed{1}$$

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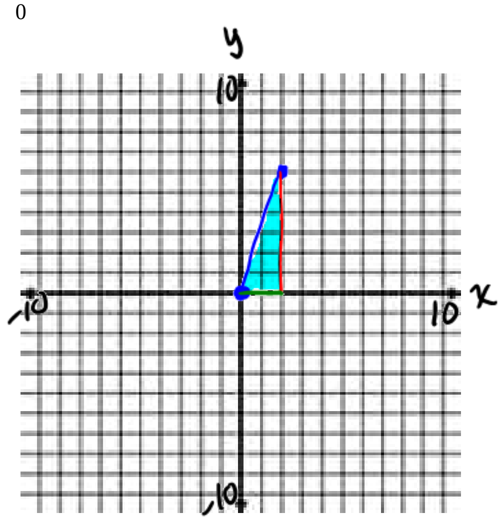
10. (5 POINTS) Sketch the region whose area is given by the definite integral. Then use a **geometric formula** to evaluate the integral. $\int_0^2 3x\,dx$.

$A=\frac{1}{2}(2)(6)$

$A=6$, since $3x\geq 0$ on $[0,2]$,

$$\int_0^2 3x\,dx = \boxed{6}$$



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11. (6 POINTS) Use **differentials** to approximate the value of the expression $\sqrt[3]{64.5}$.

$f(x+\Delta x)\approx f'(x)\,dx+f(x)$

$f(64+0.5)\approx f'(64)(0.5)+f(64)$

$$=\frac{1}{3(\sqrt[3]{64})^2}(0.5)+\sqrt[3]{64}$$

$$=\frac{1}{3\cdot 16}(0.5)+4$$

$$=\frac{1}{96}+\frac{384}{96}$$

$$=\boxed{\frac{385}{96}}$$

$f(x)=\sqrt[3]{x}=x^{1/3}$

$x=64$

$\Delta x = dx = 0.5$

$f'(x)=\frac{1}{3}x^{-2/3}$

12. (10 POINTS) Evaluate the definite integral by the **limit definition**.

$$\int_1^3 (x^2)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$a=1,\ b=3$

$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{2}{n}$

$c_i = a + i(\Delta x)$

$= 1 + \frac{2i}{n}$

$f(c_i) = \left(1+\frac{2i}{n}\right)^2$

$= 1 + \frac{4i}{n} + \frac{4i^2}{n^2}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{2}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n\overbrace{(2n+1)(n+1)}^{2n^2+3n+1}}{6}\right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{4n}{n} + \frac{4}{n} + \frac{8n^2}{3n^2} + \frac{12n}{3n^2} + \frac{4}{3n^2}\right] = \boxed{\frac{26}{3}}$$

$$= 2 + 4 + 0 + \frac{8}{3} + 0 + 0$$

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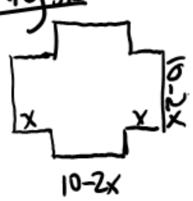
13. (10 POINTS) From a thin piece of cardboard 10 in. by 10 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? You must use calculus to solve; include your analysis, optimization, and verification—no credit awarded for trial and error! Round to the nearest tenth, if necessary.

① Analysis



② Primary Equation

$V(l,w,h) = lwh$

③ Reduce Primary to one variable

$V(x) = x(10-2x)^2$

$V(x) = 100x - 40x^2 + 4x^3$

④ Feasible domain

$x > 0,\ 10-2x > 0$

$10 > 2x$

$5 > x$

$0 < x < 5$

⑤ Optimize

Find CN

$V(x) = 100x - 40x^2 + 4x^3$

$V'(x) = 100 - 80x + 12x^2$

$0 = 4(25 - 20x + 3x^2)$

$0 = (3x-5)(x-5)$

$3x - 5 = 0$ or $x - 5 = 0$

$x = \frac{5}{3}$ or $\cancel{x=5}$

verify $x=\frac{5}{3}$ yields a max

$V''(x) = -80 + 24x$

$V''(5/3) = -80 + 24\left(\frac{5}{3}\right) = -40 < 0$

so $x = \frac{5}{3}$ yields a rel. max

$10 - 2x = 10 - 2\left(\frac{5}{3}\right)$

$= \frac{30-10}{3}$

$= \frac{20}{3}$

⑥ Conclusion

$\boxed{\left(\frac{20}{3} \times \frac{20}{3} \times \frac{5}{3}\right) in^3}$

Theorem: Summation Formulas

1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$